## A DEMONSTRATION OF MACLAURIN'S THEOREM.

BY J. S. HAYES.

[Continued from page 154, Vol. VIII.]

WE now proceed to prove that $f(x)$ is *equal* to $A+Bx+Cx^2+$&c. *ad infinitum* when all the terms of the series after $Sx^n$ have the same sign, on the condition that between the values $x=0$ and $x=x'$ $f(x)$ is neither imaginary nor infinite. This is done by proving that the limit of $L_n$, when $n$ is infinite, is zero. $L_n$ is of such a nature that it and its first $n-1$ differential coefficients vanish with $x$. It therefore must be of the form $(\varphi x)^n$ where $\varphi x$ is a function that vanishes with $x$. In fact it is proved by the ordinary methods of demonstrating Maclaurin's Theorem that $L_n$ being the remainder after $n$ terms, is equal to $\frac{x^n}{n!}f(\theta x)$ (Tod. Dif. Cal. p. 74) $=\frac{\{x\sqrt[n]{[f(\theta x)]}\}^n}{n!}$ $=(\varphi x)^n$. Now $\varphi x$ is a continuous function of $x$ and is equal to zero when $x=0$. Therefore as $x$ increases it gradually increases numerically. Of course at first it is less than 1 and greater than $-1$. While this is the case, it may be made as small as we please by sufficiently increasing $n$. Therefore, its limit is zero. If $\varphi x$ becomes greater than 1 or less than $-1$, $(\varphi x)^n$ may be made numerically as large as we please by sufficiently increasing $n$, If $\varphi x=\pm 1$, $(\varphi x)^n=\pm 1$. Therefore $(\varphi x)^n$ is either 0, infinity or $\pm 1$. when $n$ is indefinitely increased. But in the present case $L_n$ is numerically less than $L=f(x)$. Therefore it cannot be infinite. Suppose that it is $=\pm 1$ when $x=x'$. Now let $x$ change from $x_1$ to $x_2$, numerically less than $x_1$; $\varphi x$ also changes. Let the change be so small that $f(x_1)\sim f(x_2)$ $<\frac{1}{2}$ and $A+Bx_1+\ldots+Sx_1^n\sim(A+Bx_2+\ldots+Sx_2^n)<\frac{1}{2}$. Then $\varphi x$ becomes greater or less than $\pm 1$. It cannot become numerically greater, else $(\varphi x_2)^n=\infty$ when $n$ is infinite. If it becomes numerically less, $(\varphi x_2)^n$ $=0$, when $n$ is infinite. Then $f(x_1)=A+Bx_1+\ldots\pm 1$ and $f(x_2)=A$ $+Bx_2+\ldots+0$, $x_1$ being numerically greater than $x_2$ and all the signs after $Sx^n$ being the same. Then $f(x_1)\sim f(x_2)>\frac{1}{2}$. But this is contrary to the hypothesis. Therefore $\varphi x$ cannot be $\pm 1$, and $L_n$ must be zero when $n$ is infinite.

The value of the preceding demonstration (see No. 5, Vol. VIII) lies in its results. We are able, by the conditions established, to determine with reference to the convergence of a series from the series itself without reference to a remainder. When the signs change, we can also determine from the series itself within what degree of nearness the series at any required point (if convergent) approximates to the true value of the function. Some examples will be given in illustration.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}}{n}x^n + \&c.$$

By condition 2°, this is true for all values of $x$ less than unity.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \&c.$$

By condition 1°, this is true for all values of $x$ less than unity. If $x = 1$, $\log(1-x) = -\infty$. If $x > 1$, $\log(1-x)$ is imaginary.

$$\sqrt{(l^2+x)^2(l^2-x)-4lx(l^2-x)^{\frac{3}{2}}+8l^4x} = l^3 + \tfrac{5}{2}lx - \frac{5x^2}{8l} + \frac{15x^3}{48l^3} - \frac{123x^4}{384l^5} + ..$$

By condition 2°, this series is convergent to the fourth term for all values of $x$ less than $l^2$. Also it approaches to within $15x^3 \div 48l^3$ of the true value of the function.

Lagrange's and Laplace's Theorems are so dependent on Maclurin's that the conditions just established of the convergence of the latter apply equally to the former. This needs no proof.

EXAMPLES.—Given $y = z + xe^y$; expand $y$ in powers of $x$.

The result by Lagrange's Theorem is

$$y = z + xe^z + \frac{x^2}{2}2e^{2z} + \frac{x^3}{3!}3^2e^{3z} + \ldots + \frac{x^n}{n!}n^{n-1}e^{nz} + \ldots$$

By condition 1°, this furnishes a true value of $y$ for all positive values of $x$, unless, in the change from $x = 0$ to the value of $x$ under consideration, $y$ becomes infinite. But, if $x$ and $z$ are both equal to unity, the second member is an increasing series, and its sum, *ad infinitum*, is infinite. Therefore we may conclude that $y$ *does* become infinite ($z$ being $>$ or $= 1$) for some value of $x$ between zero and infinity. In this problem $\frac{dy}{dx} = e^y + xe^y\frac{dy}{dx}$, whence $\frac{dy}{dx} = \frac{e^y}{1-xe^y} = \frac{e^y}{1-xe^{z+}}$, which becomes infinite before $xe^z = 1$. This corroborates the preceding conclusion.

In Todhunter's Dif. Cal., p. 120, the following is established by the use of Lagrange's Theorem:

$$\log\frac{1-\sqrt{(1-4t)}}{2} = t + \tfrac{3}{2}t^2 + \frac{4.5}{2.3}t^3 + \frac{5.6.7}{2.3.4}4t, \&c.$$

By condition 1°, this is true for all values of $4t$ less than unity. If $4t>1$, the function is imaginary.

In the ordinary demonstration of Maclaurin's Theorem there is a condition that $f(x)$ and all its differential coefficients shall be continuous. In the demonstration here given, it is only necessary that $f(x)$ shall be continuous. This demostration therefore covers a wider field.

For example, by Maclaurin's Theorem

$$(a-x)^{\frac{1}{3}} = a^{\frac{1}{3}} - \frac{x}{3\,a^{2/3}} - \frac{1.2}{3.3}\frac{x^2}{2!a^{5/3}} - \frac{1.2.5}{3.3.3}\frac{x^3}{3!a^{7/3}} - \ldots .$$

Here $f(x)$ is continuous for all values of $x$, but its differential coefficients are not continuous between $x = 0$ and $x > a$. The ordinary demonstrations, therefore, fail for all values of $x$ greater than $a$, while this establishes the validity of the expansion for *all* values of $x$.

There is another advantage in the demonstration here given. All other methods of expansion are placed on a solid basis. In some works on the Differential Calculus it is simply *assumed* that

$$f(x) = A + Bx + Cx^2 + \ldots \text{ to infinity,}$$

$A$, $B$, $C$, &c. being unknown constants. The same assumption is made in demonstrations of the Binomial Theorem. But this assumption, it is well known in the best mathematical circles, is unwarrantable; that is, although what is assumed may be true, we have no right to assume it. But in the demonstration here given it is *proved* that $f(x) = A + Bx + Cx^2 + \ldots$ to infinity, under the conditions specified. An example will be taken from Tod. Diff. Cal., p. 91.

"Expand $\tan^{-1}x$ in powers of $x$.

Assume $\tan^{-1}x = A_0 + A_1x + A_2x^2 + \ldots A_nx^n + \ldots +$ (1)

Defferentiate both sides with respect to $x$, then

$$\frac{1}{1+x^2} = A_1 + 2A_2x + \ldots + nA_nx^{n-1} + \ldots \qquad (2)$$

But $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \ldots \qquad (3)$$

by simple division or by the binomial theorem.

Equating coefficients of like powers of $x$ in (2) and (3) we have

$$A_1 = 1,\ A_2 = 0,\ A_3 = -\tfrac{1}{3},\ A_4 = 0, \ldots$$

and putting $x = 0$ in (1) we get $A_0 = 0$; therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \text{"}$$

The fault in this solution lies in the assumption in (1). Todhunter says concerning it and others of the same kind: "We do not lay much stress upon them as exact investigations, but they may serve as exercises in differentiation." But, by the demonstration here given, the assumption in (1) is *proved* under conditions. Therefore, with it as a basis, the investigation is exact, the expression being valid by condition $2^0$ for all values of $x$ less than unity.